Lecture 2 01/16/2019

Review of Electrostatics (Cont'd)

Solution of Poisson Equation in a Volume V

Within a volume with specified non-trivial boundary conditions on Φ or $\frac{\partial \Phi}{\partial n}$, a different approach must be taken from the case in unbounded space. Let us start from the appropriate Green's function $G(\vec{x}, \vec{x}')$:

$$\nabla'^2 G(\vec{x}, \vec{x}') = -4\pi \delta^{(3)}(\vec{x} - \vec{x}')$$

We want to find the potential that satisfies the Poisson equation:

$$\nabla'^2 \Phi(\vec{x}') = \frac{-\rho(\vec{x}')}{\varepsilon_0}$$

From the two equations, we find:

$$\int_V [\Phi(\vec{x}') G(\vec{x}, \vec{x}') - G(\vec{x}, \vec{x}') \Phi(\vec{x}')] \, d\sigma' = -4\pi \int_V \delta^{(3)}(\vec{x} - \vec{x}') \Phi(\vec{x}') \, d\sigma'$$

$$+ \frac{1}{\varepsilon_0} \int_V \rho(\vec{x}') G(\vec{x}, \vec{x}') \, d\sigma' \quad (*)$$

However, recall that:

$$\Phi \nabla'^2 G - G \nabla'^2 \Phi = \vec{\nabla}' \cdot (\Phi \vec{\nabla}' G - G \vec{\nabla}' \Phi)$$

Therefore, the left-hand side of (*) is equal to:

$$\oint_S \left[\Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} - G(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} \right] da'$$

We then find:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') G(\vec{x}, \vec{x}') d\tau' - \frac{1}{4\pi} \oint_S \left[\Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} - G(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} \right] da'$$

To solve the difficulty with the second term on the right-hand side of this expression, we choose G to obey the appropriate boundary condition as follows:

(1) Dirichlet. In this case, we require that $G_D(\vec{x}, \vec{x}')|_S = 0$. This results in a unique solution:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') G_D(\vec{x}, \vec{x}') d\tau' - \frac{1}{4\pi} \oint \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'} da'$$

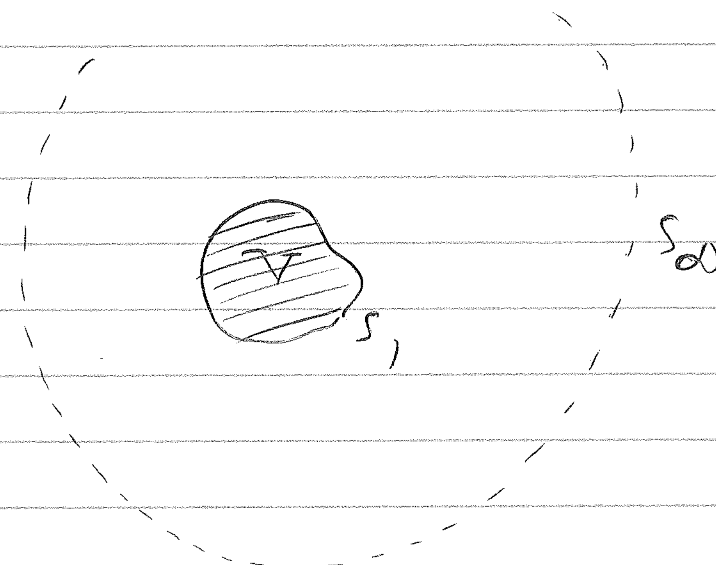
(2) Neumann. In this case, we choose $G_N(\vec{x}, \vec{x}')$ such that $\frac{\partial G(\vec{x}, \vec{x}')}{\partial n'}\Big|_S = \frac{-4\pi}{S}$. Here, S denotes the total surface

area of the boundary. We cannot set $\frac{\partial G}{\partial n'}\Big|_S$ to zero in general because of the following:

from Poisson equation

$$\int_V \nabla'^2 G \, d\tau' = \int_V \vec{\nabla}' \cdot \vec{\nabla}' G \, d\tau' = \oint \frac{\partial G}{\partial n'} \, da' \overset{\downarrow}{=} -4\pi$$

For $\frac{\partial G}{\partial n'}\Big|_S = \text{Const.}$, we find $\frac{\partial G}{\partial n'}\Big|_S = -\frac{4\pi}{S}$. This goes to zero when we consider the volume outside V as in this case $S = S_1 + S_\infty \to \infty$:



As a result, in the Neumann case we find:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') \, G_N(\vec{x}, \vec{x}') \, d\tau' + \frac{1}{4\pi} \oint_S G_N(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} \, da'$$

$$+ \frac{1}{S} \oint \Phi(\vec{x}') \, da'$$

The last term on the right-hand side is $\langle \Phi \rangle_S$. It is an additive constant up to which the Neumann problem is defined.

Some Interpretations and Notes

(1) G_D is the potential for the electrostatic problem in V that corresponds to a point charge $q = 4\pi\epsilon_0$ at $\vec{x}$ with the boundary condition $G_D|_S = 0$. I.E., the potential due to a point charge inside a volume whose boundary is grounded.

(2) Finding the solution to the Poisson equation for arbitrary $\rho(\vec{x}')$ and boundary condition on Φ or $\frac{\partial \Phi}{\partial n'}$ is reduced to finding the appropriate Green's function. For the Dirichlet problem, this is essentially a point charge problem with vanishing boundary condition.

(3) The Neumann Green's function is a slightly harder problem. However, as mentioned earlier, it is again a point charge problem with vanishing boundary condition in the case of the exterior problem.

(4) If S is a grounded conducting surface, the solution to the Dirichlet problem is:

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\, G_D(\vec{x},\vec{x}')\, d\tau'$$

We can write:

$$G_D(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|} + \tilde{G}_D(\vec{x},\vec{x}') \quad , \quad \nabla^2 \tilde{G}_D(\vec{x},\vec{x}') = 0$$

Thus:

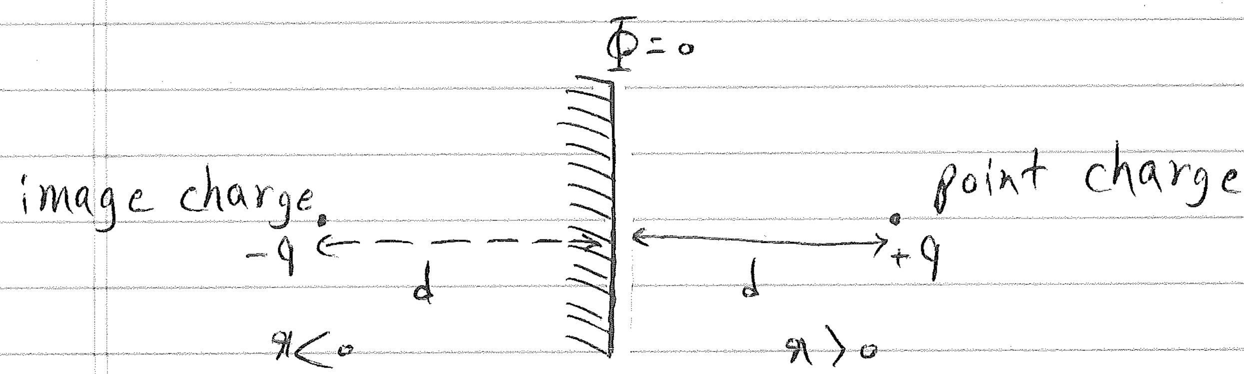
$$\Phi(\vec{x}) = \underbrace{\frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d\tau'}_{\text{Potential due to } \rho} + \underbrace{\frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{x}')\, \tilde{G}_D(\vec{x},\vec{x}')\, d\tau'}_{\text{Potential due to charges induced on the surface}}$$

(5) The Dirichlet Green's function is symmetric, i.e., $G_D(\vec{x},\vec{x}') = G_D(\vec{x}',\vec{x})$. While, $G_N(\vec{x},\vec{x}')$ can be chosen to be symmetric (reciprocity theorem).

Method of Images

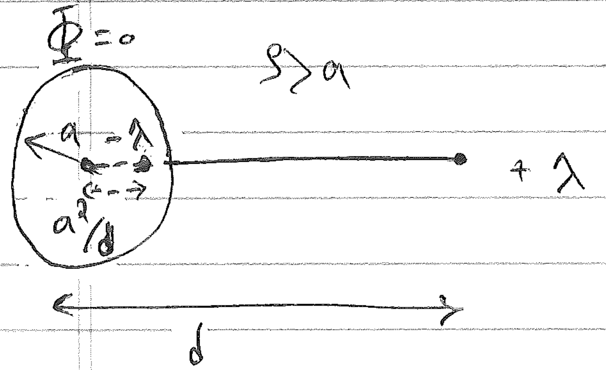
The method of images provides a powerful approach for finding the potential for simple geometries that possess some symmetry. This is particularly the case for the Dirichlet problem for planar, cylindrical, and spherical geometries. Belowe, we consider these cases separately.

(1) Plane surface. This is a half-space problem ($x \geq 0$):



For a charge $+q$ in the region $x>0$, with $\Phi=0$ at $x=0$, we can use a point charge $-q$ in the region $x<0$ located at the image of the given charge. This gives rise to the same boundary condition, and hence the same potential at $x \geq 0$ by the uniqueness of solution for a Dirichlet problem. It is important to note that Φ in the region $\underline{\underline{x<0}}$ will be different in the presence of the image charge from that in the actual system. However, as far as Φ at $x \geq 0$ is concerned, the two setups give exactly the same answer.

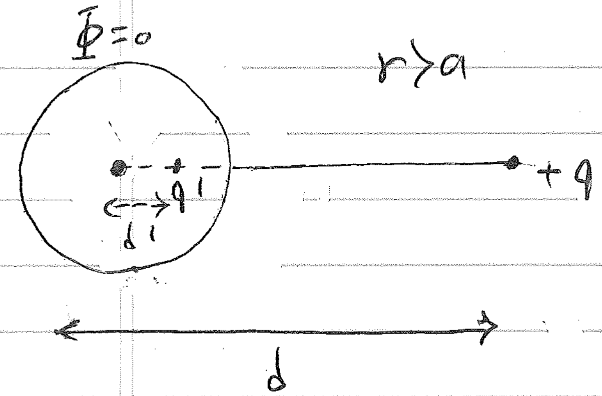
(2) Cylindrical surface and a line charge. The cross-sectional geometry of this problem is as follows:



λ: linear charge density

The image line charge gives rise to the same boundary condition, $\Phi = 0$, on the surface of the cylinder, and hence exactly the same potential at $s > a$.

(3) Spherical surface and a point charge. The cross-sectional geometry of this problem is shown below on the next page. The image charge in this case is a point



$$q' = -\frac{a}{d} q, \quad d' = \frac{a^2}{d}$$

charge situated inside the sphere. Again, the sum of the potentials from the two charges gives the correct boundary condition, as well as Φ at $r > a$.

One may consider simple variants of this problem. For example, if $\Phi = \Phi_0$ on the sphere (instead of 0), then we can add a third point charge $q'' = 4\pi\varepsilon_0 a \Phi_0$. The three charges together satisfy the required boundary condition, and hence give the correct Φ in the region $r > a$.

One may also consider the reciprocal problem where $+q$ is inside a grounded sphere. In this case, the image charge is $q' = \frac{-a}{d} q$ and is located outside the sphere at a distance $d' = \frac{a^2}{d}$

from the center of the sphere. The two charges give the required boundary condition and the correct potential anywhere <u>inside</u> the sphere in this case.

Finally, one may consider the reciprocal where $\Phi = \Phi_0$ on the sphere (instead of 0). In this case, in addition to the point charge outside the sphere, we can add a spherical surface of radius $d'' > a$ with constant surface charge density σ such that $\sigma d'' = \varepsilon_0 \Phi_0$.